Den urimelige effektivitet af matematikken i naturvidenskaben

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(1902–1995)

Nobelpris i Fysik 1963

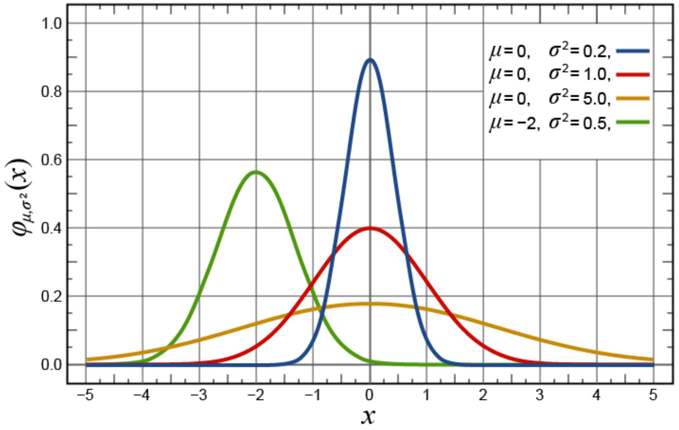
Matematikkens anvendelighed er mystisk

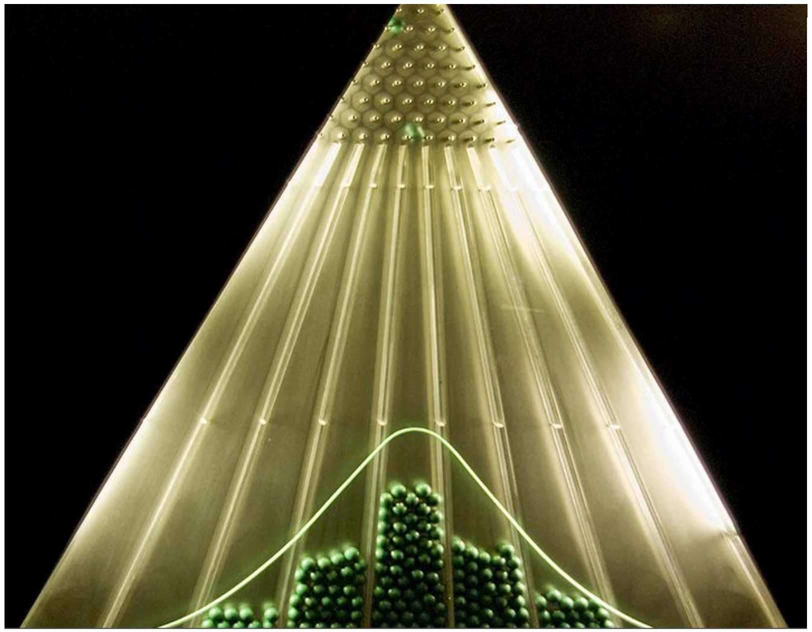
1. The first point is that mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections.
2. Secondly, just because of this circumstance, and because we do not understand the reasons of their usefulness, we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate.

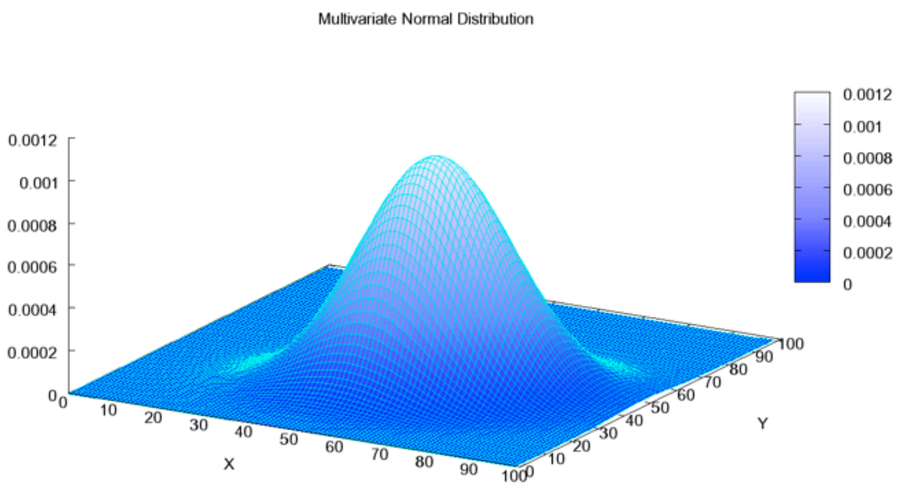
1. The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.
2. Second, it is just this uncanny usefulness of mathematical concepts that raises the question of the uniqueness of our physical theories

Normalfordelingen

$$N(\mu,\sigma^2): \quad \varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

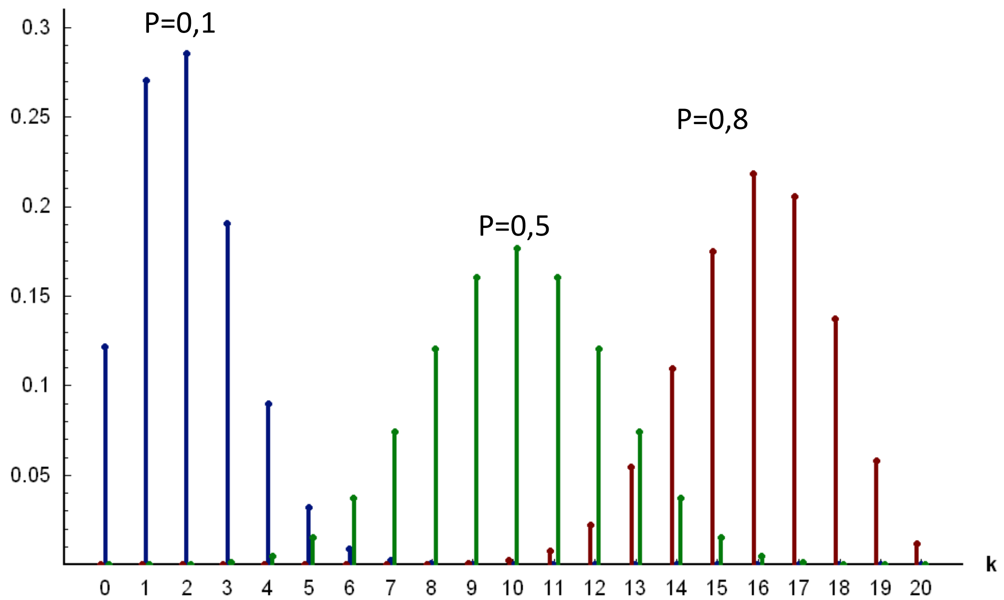


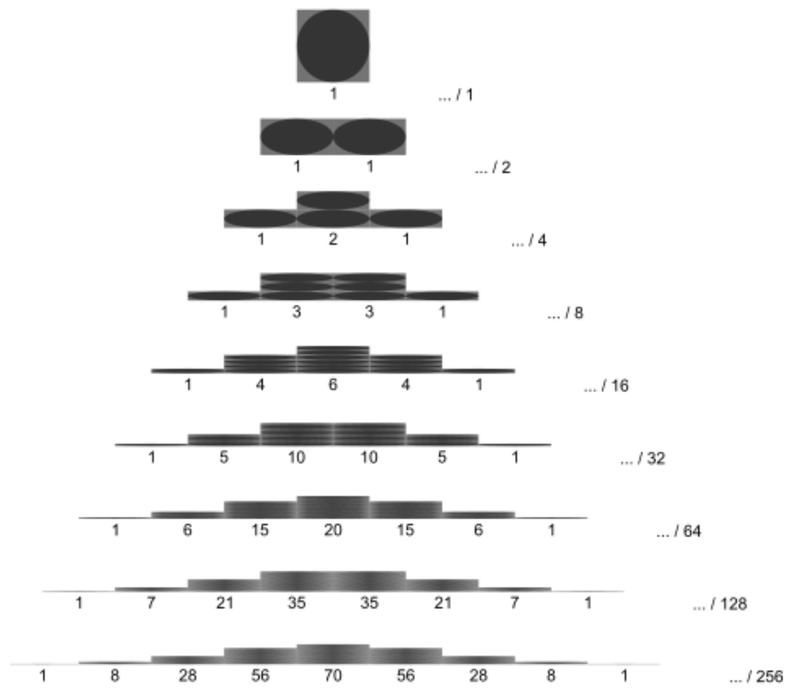




Binomialfordelingen

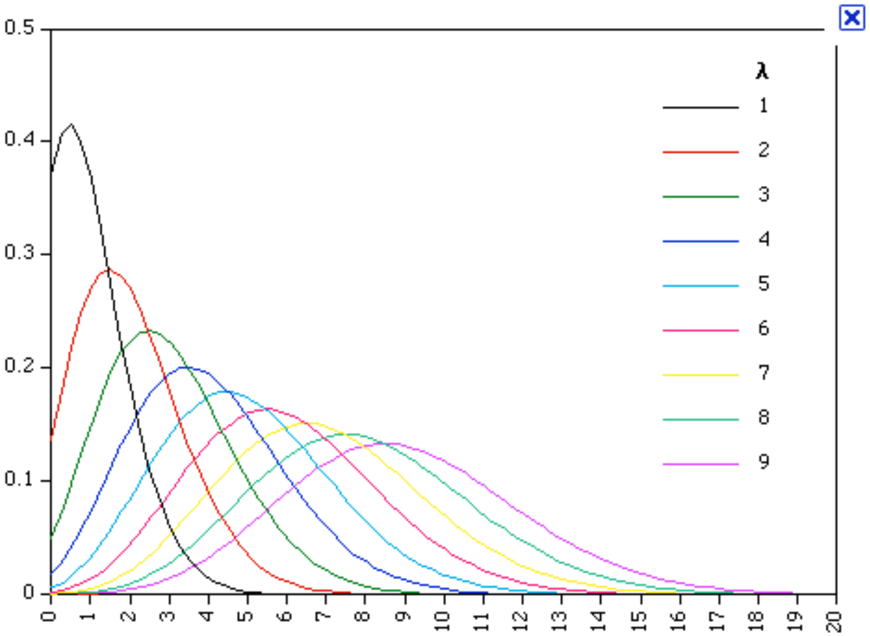
$$B(n,p)(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

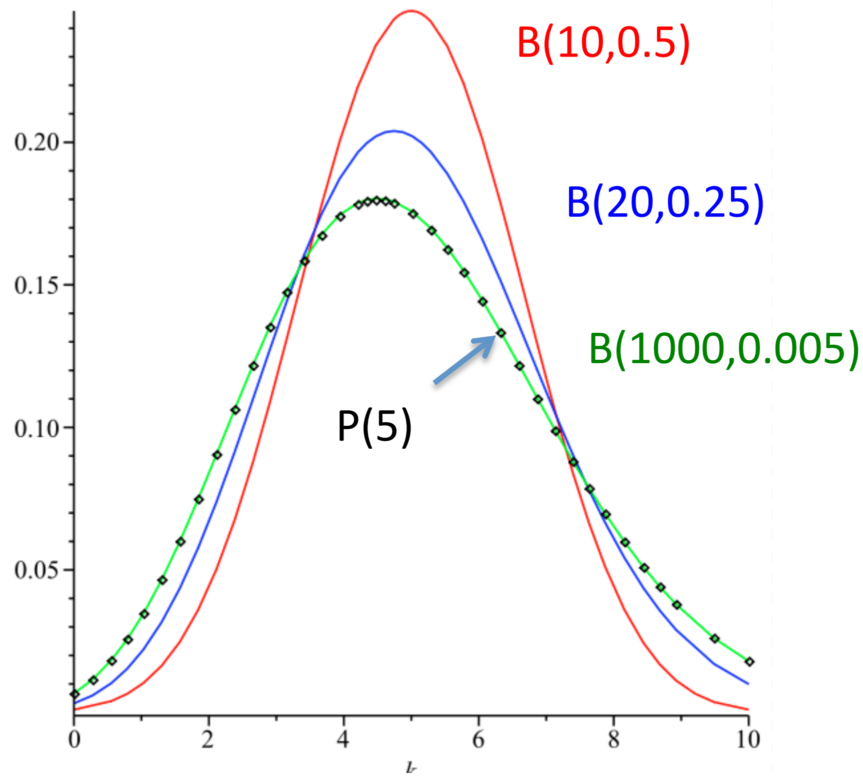




Poissonfordelingen

$$P(\lambda)(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$





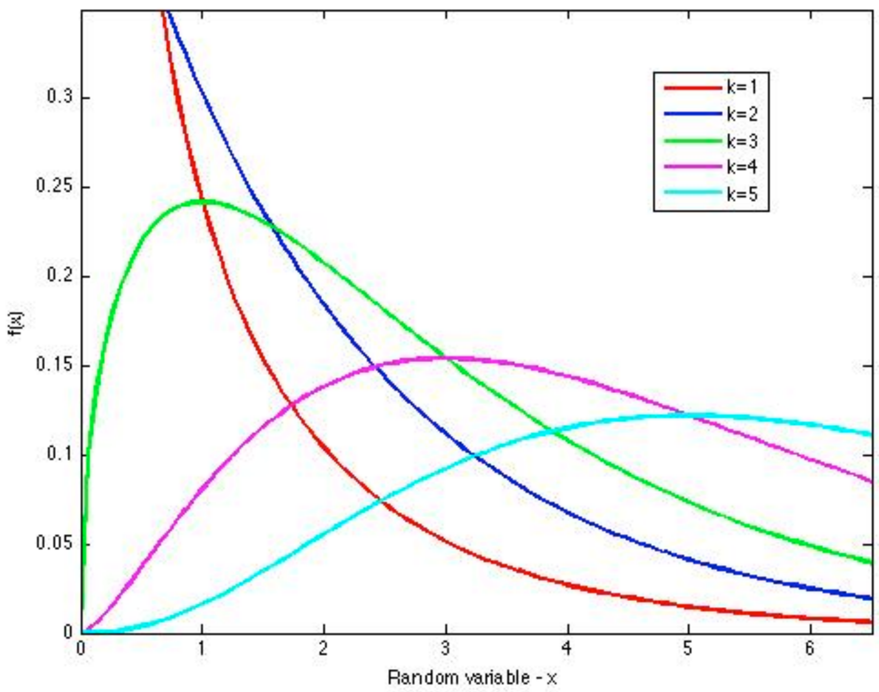
X ²(k)-fordelingen

$$\chi^2(k)(x) = \begin{cases} \dfrac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} & x \geq 0; \\ 0 & x < 0 \end{cases}$$

$$\chi^2(n-1) \sim \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

hvor X_i *er uafhængige* $N(\mu,\sigma^2)$

$$E(X^m) = k(k+2)(k+4)\cdots(k+2m-2)$$
$$= 2^m \frac{\Gamma(m+k/2)}{\Gamma(k/2)}$$



Nogle vigtige fordelinger

- **Binomialfordelingen** (B(n,p) er approximativt normalfordelt N(np,np(1-p)) for n stor og p ikke for tæt på nul.

- **Pooissonfordelingen** P(λ) er approximativt normalfordelt N(λ, λ) for λ stor.

- **X^2(k)-fordelingen** er approximativt normalfordelt N(k,2k) for k stor

$$B(n,p)(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(\lambda)(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\chi^2(k)(x) = \begin{cases} \dfrac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} & x \geq 0; \\ 0 & x < 0 \end{cases} \qquad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$$

De store tals lov og den centrale grænseværdisætning

De store tals lov

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\left(X_1 + \cdots + X_n\right) \rightarrow \mu \qquad n \rightarrow \infty$$

Konvergens kan betyde to ting:

1. Den svage lov $\quad \bar{X}_n \xrightarrow{p} \mu \quad betyder \quad \lim_{n\to\infty} P\left(\left|\bar{X}_n - \mu\right| > \varepsilon\right) = 0$

2. Den stærke lov $\quad \bar{X}_n \xrightarrow{n.o} \mu \quad betyder \quad P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 0$

Den centrale grænseværdisætning: Middelværdien af uafhængige, identisk fordelte stokastiske variable med endelige middelværdi og varians vil tilnærmeligvis være normalfordelt

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right)$$

Matematikkens Natur

… mathematics is the science of skillful operations with concepts and rules invented just for this purpose. The principal emphasis is on the invention of concepts. Mathematics would soon run out of interesting theorems if these had to be formulated in terms of the concepts which already appear in the axioms. Furthermore, whereas it is unquestionably true that the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play such an important role in physics.

… the mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms and that the concepts outside those contained in the axioms are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity.

Regulariteter i naturen

It is, as Schrodinger has remarked, a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered. One such regularity, discovered by Galileo, is that two rocks, dropped at the same time from the same height, reach the ground at the same time. The laws of nature are concerned with such regularities. Galileo's regularity is a prototype of a large class of regularities. It is a surprising regularity for three reasons.

1.The first reason that it is surprising is that it is true not only in Pisa, and in Galileo's time, it is true everywhere on the Earth, was always true, and will always be true. …

2.The second surprising feature is that the regularity which we are discussing is independent of so many conditions which could have an effect on it. …

3. As regards the present state of the world, such as the existence of the earth on which we live and on which Galileo's experiments were performed, the existence of the sun and of all our surroundings, the laws of nature are entirely silent.

Matematikkens rolle i fysikken

Mathematics, or, rather, applied mathematics, is not so much the master of the situation in this function: it is merely serving as a tool.

…

The statement that the laws of nature are written in the language of mathematics was properly made three hundred years ago;[8 It is attributed to Galileo] it is now more true than ever before.

…

It is true, of course, that physics chooses certain mathematical concepts for the formulation of the laws of nature, and surely only a fraction of all mathematical concepts is used in physics. It is true also that the concepts which were chosen were not selected arbitrarily from a listing of mathematical terms but were developed, in many if not most cases, independently by the physicist and recognized then as having been conceived before by the mathematician. It is not true, however, as is so often stated, that this had to happen because mathematics uses the simplest possible concepts and these were bound to occur in any formalism. As we saw before, the concepts of mathematics are not chosen for their conceptual simplicity even sequences of pairs of numbers are far from being the simplest concepts but for their amenability to clever manipulations and to striking, brilliant arguments.

Komplekse tal

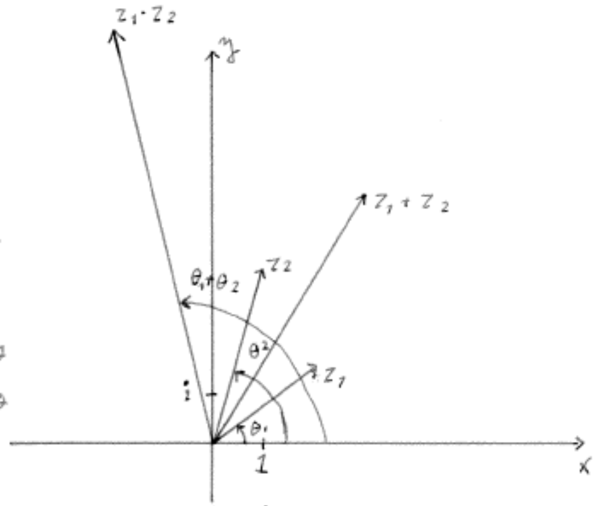
$$z = x + iy = re^{i\theta}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} y/x$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$



$$z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$$

$$z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$$

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) = r_1r_2\, e^{i(\theta_1+\theta_2)}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

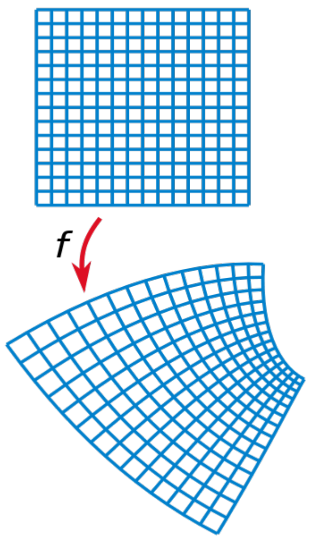
$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

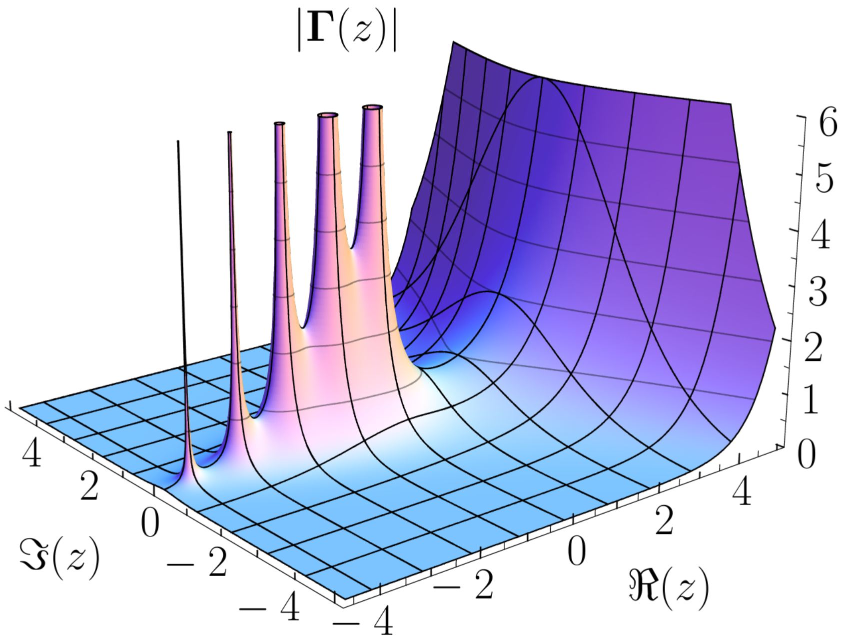
Algebraens fundamentalsætning: Ethvert ikke-konstant polynomium har en rod i den komplekse plan.

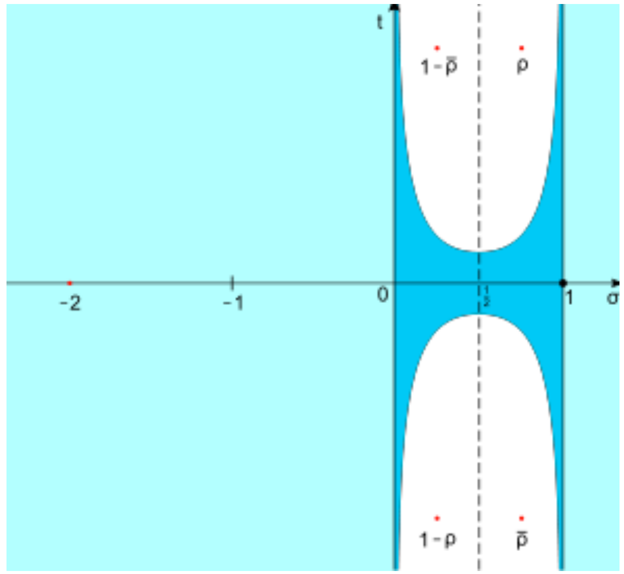
Hvis en funktion af en kompleks variabel er differentiabel, så er den vilkårligt ofte differentiabel.

Hvis f har differentialkoefficent forskellig fra 0, så bevarer f afstande og vinkler



Funktionerne Γ(z) og ζ(z)





$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t\,.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

$$\Gamma(n) = (n-1)!$$

$$\zeta(s) = 2^s \pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s),$$

Kvantemekanik

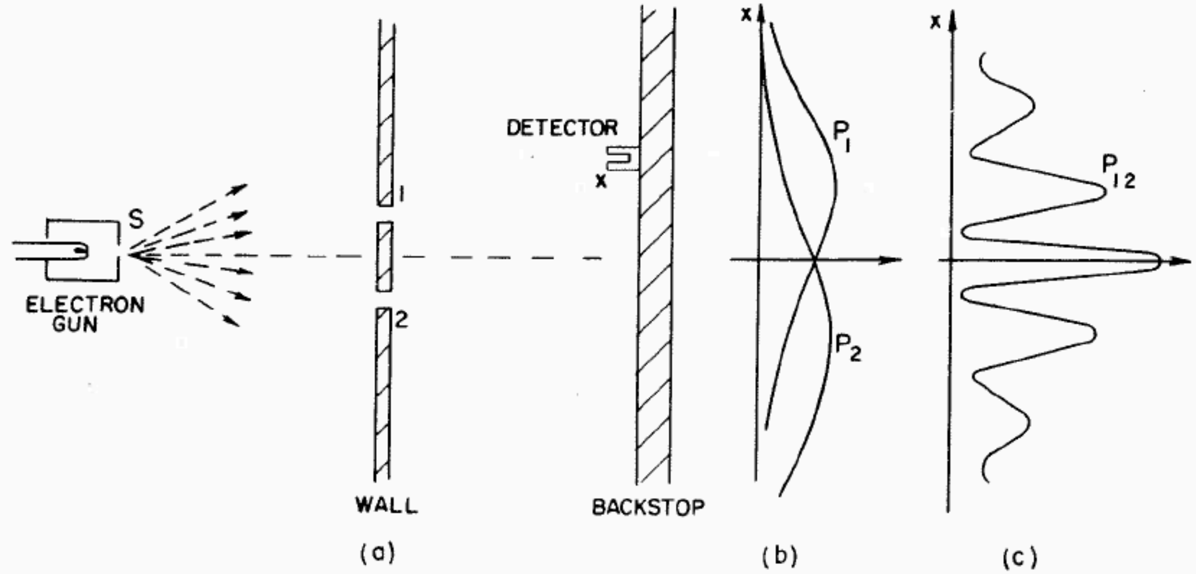
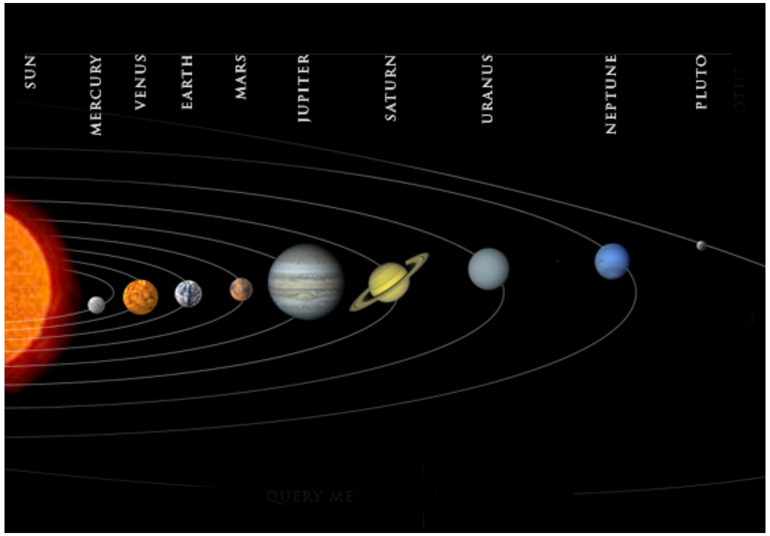


Fig 3-1. Interference experiment with electrons.

Elektroners bøle-partikel-dualisme kan beskrives med komplekse tal

$$\langle x|s\rangle = \langle x|s\rangle_1 + \langle x|s\rangle_2$$

Planetsystemet



Newtons love:

1. Inerti-loven
2. Newtons anden lov:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

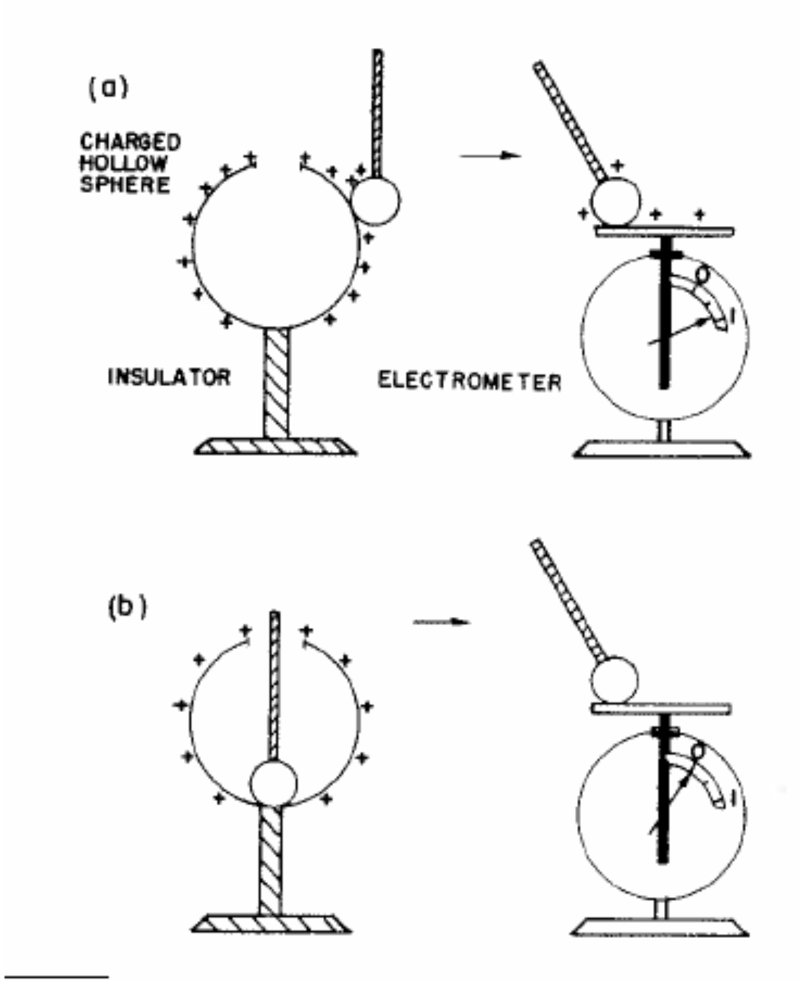
3. Loven om aktion og reaktion
4. Gravitationsloven:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\vec{e}_r$$

Let us just recapitulate our thesis on this example:

1. first, the law, particularly since a second derivative appears in it, is simple only to the mathematician, not to common sense or to non-mathematically-minded freshmen;
2. second, it is a conditional law of very limited scope. It explains nothing about the earth which attracts Galileo's rocks, or about the circular form of the moon's orbit, or about the planets of the sun. The explanation of these initial conditions is left to the geologist and the astronomer, and they have a hard time with them.

Coulombs Lov



$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{e}_r$$

Coulombs lov er eftervist ned til dimensioner 10^{-13} cm. Lamb-Retherfords målinger I 1947.

Elementær Kvantemekanik

The second example is that of ordinary, elementary quantum mechanics. This originated when Max Born noticed that some rules of computation, given by Heisenberg, were formally identical with the rules of computation with matrices, established a long time before by mathematicians. Born, Jordan, and Heisenberg then proposed to replace by matrices the position and momentum variables of the equations of classical mechanics.

…

The mathematical formalism was too dear and unchangeable so that, had the miracle of helium which was mentioned before not occurred, a true crisis would have arisen. Surely, physics would have overcome that crisis in one way or another. It is true, on the other hand, that physics as we know it today would not be possible without a constant recurrence of miracles similar to the one of the helium atom, which is perhaps the most striking miracle that has occurred in the course of the development of elementary quantum mechanics, but by far not the only one.

The empirical law of epistemology

… the "laws of nature" being of almost **fantastic accuracy** but of **strictly limited scope**. I propose to refer to the observation which these examples illustrate as the empirical law of epistemology. **Together with the laws of invariance of physical theories, it is an indispensable foundation of these theories**. Without the laws of invariance the physical theories could have been given no foundation of fact; if the empirical law of epistemology were not correct, we would lack the encouragement and reassurance which are emotional necessities, without which the "laws of nature" could not have been successfully explored.

…

Every empirical law has the disquieting quality that one does not know its limitations. We have seen that there are regularities in the events in the world around us which can be formulated in terms of mathematical concepts with an uncanny accuracy. There are, on the other hand, aspects of the world concerning which we do not believe in the existence of any accurate regularities. We call these initial conditions. The question which presents itself is whether the different regularities, that is, the various laws of nature which will be discovered, will fuse into a single consistent unit, or at least asymptotically approach such a fusion. Alternatively, it is possible that there always will be some laws of nature which have nothing in common with each other.